

# Noncommutative Symmetric Differences in Orthomodular Lattices

Gerhard Dorfer<sup>1</sup>

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We deal with the following question: What is the proper way to introduce symmetric difference in orthomodular lattices? Imposing two natural conditions on this operation, six possibilities remain: the two (commutative) normal forms of the symmetric difference in Boolean algebras and four noncommutative terms. It turns out that in many respects the noncommutative forms, though more complex with respect to the lattice operations, in their properties are much nearer to the symmetric difference in Boolean algebras than the commutative terms. As application we demonstrate the usefulness of noncommutative symmetric differences in the context of congruence relations.

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**KEY WORDS:** orthomodular lattice; symmetric difference; congruence relation.

## 1. INTRODUCTION OF SYMMETRIC DIFFERENCES

The symmetric difference plays a prominent role in the theory of Boolean algebras (BA). For instance, important properties of congruence relations in BA such as permutability, regularity and uniformity of congruences follow mainly from the fact that the symmetric difference is an associative, cancellative and invertible term function. We will recall all these notions later in detail when we deal with it.

Thus it is a manifest task to investigate symmetric difference in the more general framework of orthomodular lattices (OML). Some work in this direction can be found in (Dorfer *et al.*, 1996) and (Länger, 1998).

An orthomodular lattice  $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$  is a bounded lattice  $(L, \vee, \wedge, 0, 1)$  with an orthocomplementation  $'$ , i.e., for all  $x, y \in L$

$$x \wedge x' = 0, \quad x \vee x' = 1, \quad x'' = x, \quad x \leq y \text{ implies } y' \leq x',$$

and  $\mathcal{L}$  satisfies the orthomodular law:

$$x \leq y \text{ implies } y = x \vee (y \wedge x').$$

<sup>1</sup>Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstr. 8-10/104, 1040 Vienna, Austria; e-mail: g.dorfer@tuwien.ac.at.

As distinguished from Boolean algebras orthomodular lattices are not distributive. The following two relations provide some kind of a measure for nondistributivity in a particular OML:

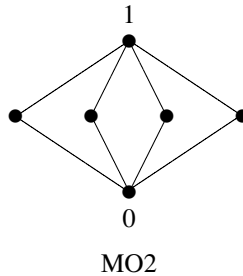
- The *commutativity relation*  $C$ :  $aCb$  if and only if the subalgebra generated by  $\{a, b\}$  in  $\mathcal{L}$  is Boolean. For instance,  $a \leq b$  or  $a \leq b'$  implies  $aCb$ .
- The *perspectivity relation*  $\sim$ :  $a \sim b$  if and only if  $a$  and  $b$  have a common (algebraic) complement, i.e. there exists an element  $c \in L$  such that

$$a \wedge c = b \wedge c = 0, \quad a \vee c = b \vee c = 1.$$

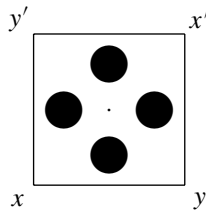
An OML  $\mathcal{L}$  is a BA if and only if  $C$  is the all relation or, equivalently, if and only if  $\sim$  is the identity.

For a solid introduction to the theory of OML we refer to (Kalmbach, 1983).

We recall a smart technique of Navara (1997) to represent elements and simplify computations in the free OML  $\mathcal{F}(x, y)$  with free generators  $x$  and  $y$ : Let  $c(x, y) := (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$  denote the commutator of  $x$  and  $y$ . Instead of  $(c(x, y))'$  we simply write  $c'(x, y)$ .  $\mathcal{F}(x, y) \cong [0, c(x, y)] \times [0, c'(x, y)]$ , where  $[0, c(x, y)] \cong 2^4$  is the 16-element BA (which is the free BA generated by two elements) with atoms  $x \wedge y, x \wedge y', x' \wedge y, x' \wedge y'$ , and  $[0, c'(x, y)] \cong \text{MO2}$  is the six-element OML with atoms  $x \wedge c'(x, y), y \wedge c'(x, y), x' \wedge c'(x, y), y' \wedge c'(x, y)$ .



The representation of elements in  $\mathcal{F}(x, y)$  refers to the following scheme:



The discs correspond to the Boolean part and the bars to the MO2-part of  $\mathcal{F}(x, y)$ . Full/empty discs refer to the presence/absence of the corresponding atoms in the Boolean part and corner angles represent the atoms in the MO2-part. While we deal

with  $\mathcal{F}(x, y)$ , deviating from (Navara, 1997), we do not indicate the generators  $x$  and  $y$  separately. In the representation one just has to remember that  $x$  is the element down left,  $y$  down right and complements are vis-à-vis. For instance,

$$(x \wedge y') \vee (x' \wedge y) = \begin{matrix} \circ & \circ \\ \bullet & \bullet \end{matrix}, \quad x' \wedge c'(x, y) = \begin{matrix} \circ & \circ \\ \bullet & \bullet \end{matrix}, \quad 1 = \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}.$$

Computations in  $\mathcal{F}(x, y)$  decompose into a Boolean part with set-theoretical operations on the discs, and a MO2-part with operations on the corner angles following the evaluation rules in MO2. For instance,

$$x = (x \wedge c'(x, y)) \vee ((x \wedge y) \vee (x \wedge y')) = \begin{matrix} \circ & \circ \\ \bullet & \bullet \end{matrix} \vee \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix} = \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix},$$

$$x' \vee y = (\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix})' \vee \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix} = \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix} \vee \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix} = \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix} = (\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix})' = (x \wedge y')'.$$

In an attempt to adopt symmetric difference for OML the first striking thing is that two different terms representing the symmetric difference in a BA may differ when they are evaluated in an OML. In particular, consider the disjunctive and conjunctive normal form

$$x \nabla y := (x \wedge y') \vee (x' \wedge y),$$

$$x \Delta y := (x \vee y) \wedge (x' \vee y').$$

Applying these operations in MO2 with generating elements  $a, b$ , we obtain

$$a \nabla b = a \nabla b' = a' \nabla b = a' \nabla b' = 0,$$

$$a \Delta b = a \Delta b'$$

$$= a' \Delta b = a' \Delta b' = 1.$$

In fact the difference between these two operations could not be larger.

First of all we have to make clear what we understand by a symmetric difference. We impose the following two natural conditions.

*Definition 1.* A binary operation  $+$  in an OML  $\mathcal{L}$  is called *symmetric difference* if

- (i)  $+$  is a *term function*,
- (ii)  $+$  coincides with the conventional symmetric difference if  $\mathcal{L}$  is a BA.

The first objective is to find out how many such operations exist.

**Theorem 2.** *For OML there are exactly six possibilities to define an operation such that (i) and (ii) are satisfied:*

$$x \nabla y = (x \wedge y') \vee (x' \wedge y)$$

$$x \Delta y = (x \vee y) \wedge (x' \vee y')$$

$$x +_l y := (x \vee (x' \wedge y)) \wedge (x' \vee y')$$

$$x +_r y := ((x \wedge y') \vee y) \wedge (x' \vee y')$$

$$x +_{l'} y := (x \vee y) \wedge (x' \vee (x \wedge y'))$$

$$x +_{r'} y := (x \vee y) \wedge ((x' \wedge y) \vee y')$$

**Proof:** Let  $\mathcal{F}_{BA}(u, v)$  denote the free BA with free generators  $u$  and  $v$ . We consider the homomorphism

$$\varphi : \begin{cases} \mathcal{F}(x, y) \rightarrow \mathcal{F}_{BA}(u, v) \\ x \mapsto u \\ y \mapsto v \end{cases}$$

Condition (i) and (ii) in Definition 1. exactly mean that the symmetric differences are given by the terms in  $\varphi^{-1}(u \Delta v)$ . Using Navara’s technique these elements have the following representation:

$$\begin{aligned} \overset{\circ}{\circ} &= (x \wedge y') \vee (x' \wedge y) = x \nabla y \\ \boxed{\overset{\circ}{\circ}} &= ((x \wedge y') \vee (x' \wedge y)) \vee c'(x, y) = (x \vee y) \wedge (x' \vee y') = x \Delta y \\ \boxed{\overset{\circ}{\circ}} &= ((x \wedge y') \vee (x' \wedge y)) \vee (x \wedge c'(x, y)) = (x \vee (x' \wedge y)) \wedge (x' \vee y') \\ \boxed{\overset{\circ}{\circ}} &= ((x \wedge y') \vee (x' \wedge y)) \vee (y \wedge c'(x, y)) = ((x \wedge y') \vee y) \wedge (x' \vee y') \\ \boxed{\overset{\circ}{\circ}} &= ((x \wedge y') \vee (x' \wedge y)) \vee (x' \wedge c'(x, y)) = (x \vee y) \wedge (x' \vee (x \wedge y')) \\ \boxed{\overset{\circ}{\circ}} &= ((x \wedge y') \vee (x' \wedge y)) \vee (y' \wedge c'(x, y)) = (x \vee y) \wedge ((x' \wedge y) \vee y') \end{aligned}$$

□

To point up the difference between the six terms we rewrite two of them in Navara’s notation:

$$x +_l y = \boxed{\overset{\circ}{\circ}} +_l \overset{\circ}{\circ} = \boxed{\overset{\circ}{\circ}}, \quad x \Delta y = \boxed{\overset{\circ}{\circ} \Delta \overset{\circ}{\circ}} = \boxed{\overset{\circ}{\circ}}.$$

In the Boolean part in both cases the conventional symmetric difference is formed, in the MO2-part the arguments do not commute and  $+_l$  results in the *left* argument whereas  $\Delta$  results in the join.

Next we summarize some properties of symmetric differences. We omit the proof which is straightforward.

**Proposition 3.** *In  $\mathcal{F}(x, y)$  the following holds true.*

- (i)  $x \Delta y = y \Delta x = x' \Delta y' = (x \vee y) \Delta (x \wedge y)$ ,
- (ii)  $x \nabla y = (x \Delta y)'$ ,  $x \Delta y = (x \nabla y)'$ ,
- (iii)  $x \Delta y$  and  $x \nabla y$  commute with both  $x$  and  $y$ ,
- (iv)  $x +_l y = y +_r x = x' +_l y' = y' +_r x'$ ,
- (v)  $(x +_l y)' = x' +_l y$ ,
- (vi)  $x +_l y$  commutes with  $x$  but does not commute with  $y$ .

In this proposition it becomes evident that there are strong interrelations between the six symmetric differences and that they naturally split into two

subclasses: one consisting of the commutative  $\Delta$  and  $\nabla$  and the other one containing the remaining four noncommutative terms. Operations within the same subclass behave very similar and, as will turn out, operations from different classes differ in their properties. Thus in the following we will state results for  $\Delta$  and  $+_l$  only. These results may be reformulated for the other symmetric differences by the help of Proposition 3 easily.

Let in the following  $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$  denote an arbitrary OML and  $a, b, c$  elements of  $L$ . The next proposition will clarify to what extent the six symmetric differences may differ.

**Proposition 4.**

- (i) *If  $a$  commutes with  $b$  then all six symmetric differences of  $a$  and  $b$  are equal.*
- (ii) *If  $a$  does not commute with  $b$  then the six symmetric differences of  $a$  and  $b$  are pairwise different.*

**Proof:** In  $\mathcal{F}(x, y)$  the six symmetric differences of  $x$  and  $y$  form an interval isomorphic to MO2, in  $\mathcal{L}$  the symmetric differences of  $a$  and  $b$  thus form a homomorphic image of MO2. Since MO2 is simple, this image either consists of one element, which is the case if  $a$  commutes with  $b$ , or is isomorphic to MO2 if  $a$  does not commute with  $b$ . □

**Corollary 5.**

- (i) *If two distinct symmetric differences coincide on the whole of  $\mathcal{L}$  then  $\mathcal{L}$  is a BA.*
- (ii) *If  $+_l$  is commutative then  $\mathcal{L}$  is a BA.*

**CANCELLATIVITY, INVERTIBILITY, AND ASSOCIATIVITY OF SYMMETRIC DIFFERENCES**

In this section we study symmetric differences in OML with respect to important properties the symmetric difference fulfils in a BA.

For the convenience of the reader we recall some basic notions from algebra. A binary operation  $\circ$  on a set  $A$  is called right cancellative (left invertible) if for arbitrary  $a, b \in A$  the equation  $x \circ a = b$  has at most (at least) one solution  $x \in A$ . Left cancellativity (right invertibility) is defined accordingly. An operation is cancellative (invertible) if it is both left and right cancellative (left and right invertible).

Firstly we recall some known results (cf. Dorfer *et al.*, 1996, Proposition 3.4, Lemma 3.6).

**Proposition 6.**

- (i) *Two elements  $a$  and  $b$  are perspective to each other if and only if there exists  $c$  such that  $a \Delta c = b \Delta c$ .*
- (ii) *Two elements  $a$  and  $b$  commute if and only if there exists an element  $c$  such that  $a \Delta c = b$ .*

By (i) we may interpret the relation of perspectivity in an OML as a measure for how far the symmetric difference  $\Delta$  is from being cancellative, and by (ii) the same applies to the complement of the commutativity relation with respect to invertibility of  $\Delta$ .

**Theorem 7.** *In the variety of OML the symmetric difference  $+_l$  satisfies the identity*

$$(x +_l y) +_l y = x.$$

**Corollary 8.**

- (i) *The symmetric difference  $+_l$  is right cancellative, i.e., if  $a +_l b = c +_l b$  then  $a = c$ .*
- (ii)  *$+_l$  is left invertible, i.e., for all  $a$  and  $b$  there exists  $c$  such that  $c +_l a = b$ .*

**Proof:** Again we apply Navara’s technique to verify the identity:

$$(x +_l y) +_l y = (\begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{smallmatrix} +_l \begin{smallmatrix} \circ & \circ & \bullet \\ \bullet & \bullet & \bullet \end{smallmatrix}) +_l \begin{smallmatrix} \circ & \circ & \bullet \\ \bullet & \bullet & \bullet \end{smallmatrix} = \begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{smallmatrix} +_l \begin{smallmatrix} \circ & \circ & \bullet \\ \bullet & \bullet & \bullet \end{smallmatrix} = \begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{smallmatrix} = x.$$

The corollary now follows easily. (i): If  $a +_l b = c +_l b$  then by  $+_l$ -adding  $b$  at the right we obtain  $a = c$ .

(ii): Choose  $c = b +_l a$ . □

In comparison to Proposition 6 we characterize elementwise under which conditions left cancellation is possible and a right inverse exists for the operation  $+_l$ .

**Proposition 9.**

- (i) *In general  $+_l$  is not left cancellative, in particular  $b +_l a = b +_l c$  if and only if  $b \wedge a = b \wedge c$  and  $b' \wedge a = b' \wedge c$ .*
- (ii) *Two elements  $a$  and  $b$  commute if and only if there exists  $c$  such that  $a +_l c = b$ .*

Here we employ Navara’s technique (as it was done in (Navara, 1997)) to represent terms in two variables without assuming that these variables are free generators. As a consequence the representations may not be unique in this case.

To distinguish this from the case with free generators we indicate the generators as indices in the corresponding positions. For example in MO2 with generators  $a$  and  $b$  we have  $c(a, b) = 0$ , i.e.,  $a \overset{\circ}{\cdot} \overset{\circ}{\cdot} b = a \overset{\circ}{\circ} \overset{\circ}{\cdot} b$ .

**Proof:** (i): If  $b +_l a = b +_l c$ , i.e.,  $b \overset{\circ}{\cdot} \overset{\circ}{\cdot} a = b \overset{\circ}{\cdot} \overset{\circ}{\cdot} c$  then intersecting both sides with  $b' = b \overset{\circ}{\circ} \overset{\circ}{\cdot} a = b \overset{\circ}{\circ} \overset{\circ}{\cdot} c$  we arrive at  $b \overset{\circ}{\circ} \overset{\circ}{\cdot} a = b \overset{\circ}{\circ} \overset{\circ}{\cdot} c$ , i.e.,  $b' \wedge a = b' \wedge c$ .

Joining both sides with  $b'$  we obtain  $b \overset{\circ}{\circ} \overset{\circ}{\cdot} a = b \overset{\circ}{\circ} \overset{\circ}{\cdot} c$ , hence  $b' \vee a' = b' \vee c'$ , i.e.,  $b \wedge a = b \wedge c$ .

Conversely, if  $b \wedge a = b \wedge c$  and  $b' \wedge a = b' \wedge c$  then immediately  $b +_l a = b +_l c$  follows.

(ii) is obvious. □

**Corollary 10.** *If a symmetric difference is cancellative or invertible on  $\mathcal{L}$  then  $\mathcal{L}$  is a BA.*

In this context we recall (Dorfer *et al.*, 1996, Theorem 3.8).

**Proposition 11.** *In the variety of OML there does not exist a binary term inducing a cancellative, respectively invertible, term function on every OML.*

As next step we address associativity of symmetric differences in OML.

**Proposition 12.** *The following are equivalent:*

- (i)  $a$  commutes with  $b$ ,
- (ii)  $(a \Delta b) \Delta b = a \Delta (b \Delta b)$ ,
- (iii)  $a +_l (a +_l b) = (a +_l a) +_l b$ .

**Proof:** Evidently (i) implies (ii) and (iii).

(ii) implies (i): Simplifying the left hand side we get

$$(a \Delta b) \Delta b = a \overset{\circ}{\circ} \overset{\circ}{\cdot} b \Delta a \overset{\circ}{\cdot} \overset{\circ}{\cdot} b = a \overset{\circ}{\cdot} \overset{\circ}{\cdot} b.$$

The right hand side is  $a \Delta (b \Delta b) = a = a \overset{\circ}{\cdot} \overset{\circ}{\cdot} b$ . Obviously  $a \overset{\circ}{\cdot} \overset{\circ}{\cdot} b = a \overset{\circ}{\circ} \overset{\circ}{\cdot} b$  implies that  $a$  commutes with  $b$ .

A similar argument yields that (iii) implies (i). □

We see that the associative law is valid for symmetric differences in OML in some special situations only. The following corollary is derived easily.

**Corollary 13.** *If a symmetric difference is associative in  $\mathcal{L}$  then  $\mathcal{L}$  is a BA.*

We also provide a positive result where again the discrepancy among symmetric differences appears.

**Proposition 14.** *If  $b$  commutes with  $a$  and  $c$  then*

- (i)  $(a \Delta b) \Delta c = a \Delta (b \Delta c)$ ,
- (ii)  $(b +_l a) +_l c = b +_l (a +_l c)$ .

**Proof:** We proof (ii) in detail, (i) follows similarly.

Since  $b$  commutes with  $a$  and  $c$  all computations occur in the commutator  $C(b) \cong [0, b] \times [0, b']$  (see e.g. Kalmbach, 1983, 1.3.1. Theorem). Therefore it is sufficient to verify the assertion for  $b = 0$  and  $b = 1$ .

For  $b = 0$  we have

$$(0 +_l a) +_l c = a +_l c = 0 +_l (a +_l c),$$

and for  $b = 1$  due to Proposition 3

$$(1 +_l a) +_l c = a' +_l c = (a +_l c)' = 1 +_l (a +_l c).$$

□

Without going into detail we mention that also the distributivity of the meet operation with respect to the symmetric difference cannot be generalized from BA to OML even if one additionally considers also all *possible* meet operations (in the sense of Definition 1).

### 3. CONGRUENCE RELATIONS

It is well known that there is a bijection between congruence relations of an OML  $\mathcal{L}$  and certain ideals of  $\mathcal{L}$ , so-called p-ideals (Finch, 1996; or orthomodular ideal, see Kalmbach, 1983): A lattice ideal  $I$  is a p-ideal if it is closed under perspectivity, i.e.,  $a \in I$  and  $b \sim a$  imply  $b \in I$ . For a congruence  $\theta$  on  $\mathcal{L}$  and  $a \in L$  let  $[a]\theta$  denote the congruence class of  $a$ . In the following theorem the relationship between congruences and p-ideals is summarized.

**Theorem 15.**

- (i)  $I = [0]\theta$  for some congruence  $\theta$  if and only if  $I$  is a p-ideal.
- (ii) A lattice ideal  $I$  is a p-ideal if and only if for all  $x$  in  $L$

$$x \wedge (I \vee x') \subseteq I$$

(where  $x \wedge (I \vee x') = \{x \wedge (i \vee x') \mid i \in I\}$ ).

- (iii) For a p-ideal  $I$  the congruence  $\theta$  corresponding to  $I$  is given by the condition  $x\theta y$  if and only if  $x \Delta y \in I$ .



Using a noncommutative symmetric difference this result can be modified as follows.

**Proposition 16.**

(ii') A lattice ideal  $I$  is a  $p$ -ideal if and only if for all  $x$  in  $L$

$$x +_l (I +_l x) \subseteq I.$$

(iii') For a  $p$ -ideal  $I$  the congruence  $\theta$  corresponding to  $I$  is given by the condition  $x\theta y$  if and only if  $x +_l y \in I$ .

**Proof:** (ii'): We compare the elements  $x \wedge (i \vee x')$  and  $x +_l (i +_l x)$  for  $x$  in  $L$  and  $i$  in  $I$ .

$$\begin{aligned} x \wedge (i \vee x') &= x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} \wedge \left( x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} \vee x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} \right) \\ &= x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} \wedge x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} \\ &= x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array}, \\ x +_l (i +_l x) &= x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} +_l \left( x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} +_l x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} \right) \\ &= x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} +_l x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array} \\ &= x \begin{array}{|c|} \hline \circ \\ \hline \bullet \\ \hline \circ \\ \hline \end{array}. \end{aligned}$$

This means that  $x +_l (i +_l x) = (x \wedge (i \vee x')) \vee (i \wedge x')$ . Since  $i \wedge x'$  is in  $I$  ( $I$  is an order ideal) this means that the conditions in (ii) and (ii') are equivalent.

(iii'): We have to show that for a  $p$ -ideal  $I$  the condition  $x \Delta y \in I$  is equivalent to  $x +_l y \in I$ . Since  $x +_l y \leq x \Delta y$  one direction is clear. On the other hand, if  $x +_l y \in I$  then  $y +_l x = y +_l ((x +_l y) +_l y)$  is also in  $I$  (see (ii')), and  $x \Delta y = (x +_l y) \vee (y +_l x)$ . □

*Remark 1.* We want to point out that using noncommutative symmetric differences we can state the requirements for a congruence kernel ( $p$ -ideal) of an OML very similar to groups/Boolean algebras. One can even push this similarity further:

**Proposition 17.** A subset  $I$  of  $L$  is a congruence kernel if and only if

- (i)  $(I, +_l, 0)$  is a subalgebra of  $(L, +_l, 0)$  (subgroup condition),
- (ii)  $x +_l (I +_l x) \subseteq I$  for all  $x \in L$  (normal subgroup condition),
- (iii)  $I$  is an order ideal in  $(L, \leq)$  (as in BA).

Though these conditions seem very natural one has to be careful: for instance, the second condition may *not* be substituted by  $(x +_l I) +_l x \subseteq I$  for all  $x \in L$ . We leave the proof to the interested reader.

Now we turn towards congruence classes. We recall some notions from universal algebra. An algebra is called

- *congruence regular* if, for any congruence of this algebra, every congruence class determines the whole congruence uniquely,
- *congruence uniform* if for any fixed congruence all congruence classes have the same cardinality,
- *congruence permutable* if any two congruences  $\theta$  and  $\phi$  permute, i.e.,

$$\theta \circ \phi = \phi \circ \theta (= \theta \vee \phi).$$

A variety of algebras is called congruence regular (uniform, permutable) if all algebras of this variety have this property. For varieties congruence permutability and congruence regularity can be characterized by so-called Mal'cev conditions (cf. Csákány, 1970; Mal'cev, 1954).

**Theorem 18.**

- (i) *A variety is congruence permutable if and only if there exists a ternary term  $m(x, y, z)$  such that the identities*

$$m(x, z, z) = x, \quad m(x, x, z) = z$$

*hold true in the variety.  $m(x, y, z)$  is called Mal'cev term for this variety.*

- (ii) *A variety is congruence regular if and only if there exist ternary terms  $t_1(x, y, z), \dots, t_n(x, y, z)$  such that the condition*

$$[t_1(x, y, z) = z, \dots, t_n(x, y, z) = z] \quad \text{if and only if} \quad x = y$$

*is fulfilled in the variety.  $t_1(x, y, z), \dots, t_n(x, y, z)$  are called Csákány terms for this variety.*

**Theorem 19.** *In the variety of OML the term*

$$(x +_l y) +_l z$$

*is a Mal'cev and a Csákány term.*

**Proof:**  $(x +_l y) +_l z$  is a Mal'cev term: By Theorem 7 we have

$$(x +_l z) +_l z = x,$$

and obviously

$$(x +_l x) +_l z = 0 +_l z = z.$$

$(x +_l y) +_l z$  is a Csákány term: The *if* part of the condition in Theorem 18 (ii) was verified just before. Now, suppose  $(x +_l y) +_l z = z$ , then by  $+_l$ -adding  $z$  from the right side we get  $x +_l y = 0$  and by adding  $y$  we arrive at  $x = y$ .  $\square$

Let  $I$  be a  $p$ -ideal,  $\theta$  the congruence relation corresponding to  $I$  and  $a$  in  $L$ . In Dorfer (2001, Proposition 3.2, 3.3) the following relationship between the congruence classes of  $\theta$  was given:

$$[a]\theta = a \vee (I \wedge a'), \quad I = [a]\theta \Delta [a]\theta.$$

Using non-commutative symmetric differences much simpler (group-like) formulas occur.

**Theorem 20.** *If  $\theta$  is a congruence in  $\mathcal{L}$ ,  $I = [0]\theta$  and  $a \in L$  then*

- (i)  $[a]\theta = I +_l a$ ,
- (ii)  $I = [a]\theta +_l a$ .

*In particular, these formulas provide bijections between the congruence classes.*

**Proof:** (i): If  $i \in I$  then  $(i +_l a)\theta(0 +_l a) = a$ . Conversely, if  $x \in [a]\theta$  then  $(x +_l a)\theta(a +_l a) = 0$ , hence  $x +_l a \in I$  and by Theorem 7  $x = (x +_l a) +_l a \in I +_l a$ .

(ii):  $[a]\theta +_l a = (I +_l a) +_l a = I$ . The concluding assertion follows easily. □

We should mention that with commutative symmetric differences no such simple formulas can be found. A similar phenomenon appears when terms representing implication in OML are investigated (cf. Hajda *et al.*, 2001).

Finally we repeat a result (cf. Dorfer, 2001, Theorem 4.1–4.3), which now is an immediate consequence of Theorem 19 and Theorem 20.

**Corollary 21.** *The variety of OML is congruence regular, uniform and permutable.*

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### REFERENCES

Hajda, I., Halaš, R. and Länger, H. (2001). Orthomodular implication algebras. *International Journal of Theoretical Physics* **40**, 1875–1884.

Csákány, B. (1970). Characterizations of regular varieties. *Acta Scientiarum Mathematicarum* (Szeged) **31**, 187–189.

Dorfer, G., Dvurečenskij, A. and Länger, H. (1996). Symmetric difference in orthomodular lattices. *Mathematical Slovaca* **46**, 435–444.

- Dorfer, G. (2001). Some properties of congruence relations in orthomodular lattices. *Discussiones Mathematicae General Algebra and Applications* **21**, 57–66.
- Finch, P. D. (1966). Congruence relations on orthomodular lattices. *Journal of the Australian Mathematical Society* **6**, 46–54.
- Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, London.
- Länger, H. (1998). Generalizations of the correspondence between Boolean algebras and Boolean rings to orthomodular lattices. *Tatra Mountains Mathematical Publications* **15**, 97–105.
- Mal'cev, A. I. (1954). On the general theory of algebraic systems (Russian). *Matematicheskij Sbornik* **35**, 3–20.
- Navara, M. (1997). On generating finite orthomodular sublattices. *Tatra Mountains Mathematical Publications* **10**, 109–117.